Calculating Multi-Element Antennas Using Mathcad®

Here is a technique for mathematical analysis of antennas, using a Yagi-Uda array of several elements as an example

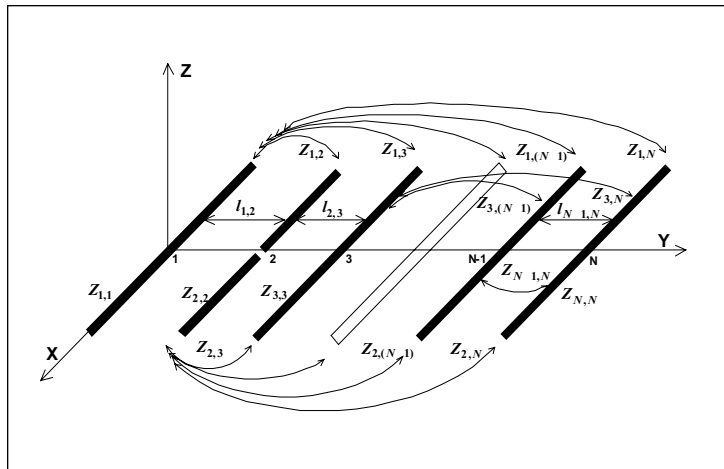**By Néstor E. Arias**
National University of Tucumán

A linear antenna can be combined with another to obtain a new antenna with more gain. If there are more than two or three elements used to build that antenna, estimating the proper and mutual impedances, excitation point and gain becomes complicated.

Estimation of these variables involves the use of exponential integrals, and the number of calculations necessary to form the matrix impedance is equal to the number of elements taken by pairs. For example, in the case of 15 elements, it is necessary to carry out 105 calculations plus 15 more of their proper impedances by means of the integrals as those mentioned above. It is also necessary to estimate the corresponding currents for matrix calculation, which is essential for calculating the gain. Because the number of calculations necessary to estimate the necessary elements can be numerous, and because of the exclusion of any other parameters, the use of a mathematical program is ideal for these calculations.

This article discusses a calculation method using the Mathcad® program (we will use Mathcad 7 Professional), which estimates and plots radiation diagrams on the three planes of space for multi-element antennas. The Mathcad program will also obtain the excitation point impedance.

Multi-element antennas

A thin linear antenna of an approximate length of $\lambda/2$ can be combined with one or more similar antennas to create a new type of antenna with a different characteristic radiation than that obtained with a single element.

▲ **Figure 1. Yagi-Uda antenna.**

Various types of multi-element antennas exist, with groups defined by the way in which the different elements are physically arranged, and how they are excited with regard to phase and amplitude.

A group of two linear antennas of an approximate length of $\lambda/2$ successively distributed in space, in parallel with one another and having a single excited element constitutes the type of antenna known as a Yagi-Uda (Figure 1). This antenna features electromagnetic coupling of all the elements of the system, which can also include nearby physical elements, such as trees, people, roofs and the ground. To simplify calculations, the antenna will be regarded in an empty space free of strange couplings, and only taking into account the antenna elements present in that region.

Array gain

It is necessary to calculate the impedance matrix $[Z]$ in order to obtain gain, the radiation patterns for each plane of space and the excitation point impedance. For each disposition of elements of the array, the matrix impedance contains the proper and mutual impedances of the system. The second element normally is excited, so a column tension matrix $[v]$ can be defined where that second element is not zero.

$$[v] = [Z] \times [I] \therefore [I] = \left([Z]^{-1} \times [v] \right) \tag{1}$$

where $I = l$solve (Z, v).

The matrix $[I]$ contains the currents of all the elements and therefore, it is possible to calculate the gain on the space planes and the excitation point impedance. The impedance is

$$Z_i = \frac{v_2}{I_2} \tag{2}$$

The array gain, relative to a half-wave dipole gain, can be calculated through the currents in each of the elements. To calculate the array gain, it is necessary to determine the actual excitation power of the array.

$$W = R_e \left[(I_2)^2 (Z_i) \right] \tag{3}$$

This power can be used, in turn, to excite a half-wave length dipole which will serve as reference, with a 73-ohm input impedance whose current is

$$i_{00} = \sqrt{\frac{W}{73}} \tag{4}$$

Using the calculated values and applying the diagram multiplication principle, the array gain in each plane is represented in Equation 5. Equations 5(a) and 5(b) (below) describe the XY and ZY planes. Gain in the XZ plane is

$$G_{plane\ XZ} = \frac{\sum_{1}^{N} I_n}{i_{00}} fd_{\frac{\lambda}{2}} \tag{5c}$$

where the array gain is relative to the half-wave dipole.

Given two antennas of lengths l_1 and l_2 separated by a distance d, as shown in Figure 2, the mutual impedance $Z_{2,1}$ can be calculated as in Equation (6). There are many references that include methods for calculation of mutual impedances between linear elements of unequal lengths. The method described by Cox [1] has been used for this paper.

By successively applying the calculation in Equation (6), all of the mutual impedances that are involved in a system of N elements are obtained. The same equation can be used to obtain the proper impedances by defining $l_1 = l_2$ and $d = 0.02$, where d represents the width of an element.

Program sequence

To develop a Mathcad program to perform these cal-

$$G_{azimuth} = \frac{I_1 e^{j0} + I_2 e^{j\beta 1_{1,2}\cos(\phi)} + ... + I_{N-1}e^{j\beta 1_{1,N-1}\cos(\phi)} + I_N e^{j\beta 1_{1,N}\cos(\phi)}}{i_{00}} fd_{\frac{\lambda}{2}}$$

where,
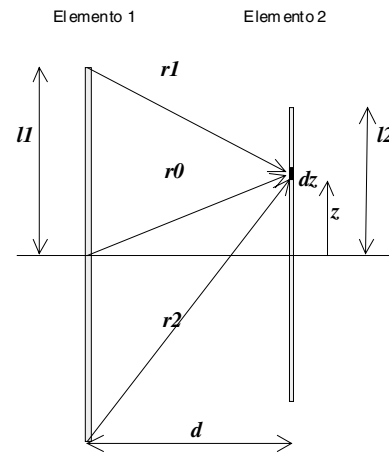
ϕ = Azimuth angle;
β = phase constant;
$l_{1,2}...l_{1,N}$ = distances of the elements to the origin;
$I_1...I_N$ = currents of the elements;
$fd_{\lambda/2}$ = diagram factor of the half-wave dipole;
N = the number of elements.

▲ **Equation 5(a). Gain in the XY plane.**

$$G_{elevation} = \frac{I_1 e^{j0} + I_2 e^{j\beta 1_{1,2}\cos(\phi)} + ... + I_{N-1}e^{j\beta 1_{1,N-1}\cos(\phi)} + I_N e^{j\beta 1_{1,N}\cos(\phi)}}{i_{00}}$$

θ = elevation angle.

▲ **Equation 5(b). Gain in the ZY plane.**

▲ **Figure 2. Two dipoles of different length.**

$$r_0 = \sqrt{d^2 + z^2}$$

$$r_1 = \sqrt{d^2 + (l_1 - z)^2}$$

$$r_2 = \sqrt{d^2 + (l_1 + z)^2}$$

$$Z_{2,1} = \frac{30}{\sin(\beta \times l_1) \times \sin(\beta \times l_2)} \left[\begin{array}{c} \int_0^{l_2} \frac{e^{-j\beta(r_1 - l_2 + z)}}{r_1} dz - \int_0^{l_2} \frac{e^{-j\beta(r_1 + l_2 - z)}}{r_1} dz + \\ \int_0^{l_2} \frac{e^{-j\beta(r_2 - l_2 + z)}}{r_2} dz - \int_0^{l_2} \frac{e^{-j\beta(r_2 + l_2 - z)}}{r_2} dz - \\ 2 \times \cos(\beta \times l_1) \int_0^{l_2} \frac{e^{-j\beta(r_0 - l_2 + z)}}{r_0} dz + 2 \times \cos(\beta \times l_1) \int_0^{l_2} \frac{e^{-j\beta(r_0 + l_2 - z)}}{r_0} dz \end{array} \right]$$

▲ **Equation (6). Calculation of mutual impedances.**

culations, the following steps have been followed:

Step 1. Necessary data
- Proper length
- Separation between elements
- Wavelength
- Number of elements *N*

Step 2.
- The program calculates a matrix (*l*) that contains the data entered, the distances of the elements to the origin and the proper longitudes.
- The limit conditions for calculation of the integrals with Equation (2).

Step 3.
- Matrix **Z** calculation.

Step 4.
- Definition of matrix *v* and calculation of the current matrix.

Step 5.
- Calculation of power W_R and current i_{oo}.

Step 6.
- Subroutine for the calculation of equations (5a), (5b) and (5c).

Step 7.
- Graphs on planes XY, YZ and XZ.

Figure 3 shows the radiation patterns relative to the half-wave dipole in a 10-element antenna for the ZY elevation and XY azimuth planes.

The Appendix shows the Mathcad programs used to perform these step-by-step calculations.

Conclusion

Any other value of *N*, length and separation between elements may be used in the Mathcad calculation. Steps in the optimization of the system to obtain maximum gain and the best situation in the forward-backward relation must be performed manually.

Author information

Nestor Arias is a faculty member at the National University of Tucumán (Universidad Nacional de Tucumán), San Miguel de Tucumán (4000), Argentina. For further information, please contact the author via e-mail at narias@herrera.unt.edu.ar.

▲ **Figure 3. Radiation patterns relative to the half-wave dipole for a 10-element antenna.**

References

1. C. Russel Cox, "Mutual Impedance Between Vertical Antennas of Anequal Heights," *Proceeding of the I..R.E.,* November 1947.

2. E.C. Jordan, K.G. Balmain *Electromagnetic Waves and Radiating Systems,* Prentice-Hall., Englewood Cliffs, NJ: 1968.

3. J.D. Kraus *Antennas,* McGraw-Hill: 1950.

4. The first element of a matrix is one. Therefore, it should be modified in Math-Option Array Origin to =1.

Appendix — Mathcad programs

$$:=1 \qquad N:=10 \qquad :=2$$

$$l_{1\ 1}:=.245 \quad l_{2\ 2}:=.235 \quad l_{3\ 3}:=.2 \quad l_{4\ 4}:=.18 \quad l_{5\ 5}:=.16 \qquad \texttt{<==Proper lengths Data}$$

$$l_{6\ 6}:=.15 \quad l_{7\ 7}:=.15 \quad l_{8\ 8}:=.15 \quad l_{9\ 9}:=.15 \quad l_{10\ 10}:=.15$$

$$l_{1\ 2}:=.17 \quad l_{2\ 3}:=.15 \quad l_{3\ 4}:=.15 \quad l_{4\ 5}:=.15 \quad l_{5\ 6}:=.15 \qquad \texttt{<== Separation of Elements Data}$$

$$l_{6\ 7}:=.21 \quad l_{7\ 8}:=.15 \quad l_{8\ 9}:=.15 \quad l_{9\ 10}:=.15$$

▲ **Step 1 of the Mathcad program.**

Subroutines set; Matrix I

$$J:=\begin{vmatrix} j \leftarrow 2 \\ \text{while } j \leq N \\ \quad \begin{vmatrix} a_{1\ j} \leftarrow 0 + l_{j-1\ j} \\ j \leftarrow j+1 \end{vmatrix} \\ a \\ i \leftarrow 2 \\ \text{while } i \leq N \\ \quad \begin{vmatrix} J_{1\ i} \leftarrow \sum_{n=2}^{i} a_{1\ n} \\ i \leftarrow 1 + i \end{vmatrix} \\ J \end{vmatrix}$$

$$U:=\begin{vmatrix} \text{for } i \quad 1..N \\ \quad \text{for } j \quad 1..N \\ \qquad U_{i\ j} \leftarrow J_{1\ j} \\ U \end{vmatrix}$$

$$UU:=\begin{vmatrix} i \leftarrow 1 \\ \text{while } i \leq N \\ \quad \begin{vmatrix} j \leftarrow 2 \\ \text{while } j \leq N \\ \quad \begin{vmatrix} UU_{i\ j} \leftarrow U_{i\ j} - U_{i\ i} \\ j \leftarrow j+1 \end{vmatrix} \\ i \leftarrow i+1 \end{vmatrix} \\ UU \end{vmatrix}$$

$$1:=Q$$

$$Q:=\begin{vmatrix} \text{for } i \quad 1..N \\ \quad \text{for } j \quad 1..N \\ \qquad Q_{i\ j} \leftarrow l_{j\ j} \text{ if } i=j \\ \qquad Q_{i\ j} \leftarrow UU_{i\ j} \text{ otherwise} \\ \qquad Q_{i\ j} \leftarrow Q_{j\ i} \text{ if } i \geq j \\ Q \end{vmatrix}$$

$$L:=\begin{vmatrix} \text{for } i \quad 1..N \\ \quad \text{for } j \quad 1..N \\ \qquad L_{i\ j} \leftarrow -l_{j\ j} \text{ if } i=j \\ \qquad L_{i\ j} \leftarrow 0 \text{ if } i \quad j \\ \qquad L_{i\ j} \leftarrow \dfrac{\left(l_{j\ j} - l_{i\ i}\right)}{2} \text{ otherwise} \\ L \end{vmatrix}$$

▲ **Step 2 of the Mathcad program.**

$$Z := \begin{aligned} &\text{for } i \in 1..N \\ &\quad \text{for } j \in 1..N \\ &\qquad L \leftarrow l_{i\,j} \quad \text{if } i = j \\ &\qquad L \leftarrow l_{j\,j} \quad \text{otherwise} \\ &\qquad LL \leftarrow l_{i\,j} \quad \text{if } i = j \\ &\qquad LL \leftarrow l_{i\,i} \quad \text{otherwise} \\ &\qquad d \leftarrow 0.02 \quad \text{if } i = j \\ &\qquad d \leftarrow l_{i\,j} \quad \text{otherwise} \end{aligned}$$

$$b_{i\,j} \leftarrow \left[\int_0^L \frac{e^{-1j\left[\sqrt{d^2+(LL-z)^2}-L+z\right]}}{\sqrt{d^2+(LL-z)^2}}\,dz - \int_0^L \frac{e^{-1j\left[\sqrt{d^2+(LL-z)^2}+L-z\right]}}{\sqrt{d^2+(LL-z)^2}}\,dz \right] \cdot \frac{30}{\sin(L+.0000001)\sin(LL+.00000001)}$$

$$c_{i\,j} \leftarrow \left[\int_0^L \frac{e^{-1j\left[\sqrt{d^2+(LL+z)^2}-L+z\right]}}{\sqrt{d^2+(LL+z)^2}}\,dz - \int_0^L \frac{e^{-1j\left[\sqrt{d^2+(LL+z)^2}+L-z\right]}}{\sqrt{d^2+(LL+z)^2}}\,dz \right] \cdot \frac{30}{\sin(L+.0000001)\sin(LL+.00000001)}$$

$$f_{i\,j} \leftarrow 2\cos(LL) \left[\int_0^L \frac{e^{-1j\left(\sqrt{d^2+z^2}+L-z\right)}}{\sqrt{d^2+z^2}}\,dz - \int_0^L \frac{e^{-1j\left(\sqrt{d^2+z^2}-L+z\right)}}{\sqrt{d^2+z^2}}\,dz \right] \cdot \frac{30}{\sin(L+.0000001)\sin(LL+.00000001)}$$

$$Z_{i\,j} \leftarrow \left(b_{i\,j}+c_{i\,j}+f_{i\,j}\right)$$
$$Z_{i\,j} \leftarrow Z_{j\,i} \quad \text{if } i \ge j$$
$$Z$$
$$Z$$

$$Z \; \begin{bmatrix}
68.718-27.3i & 50.714-11.897i & 17.767-26.218i & -4.56-21.028i & -13.17-7.22i & -10.714-4.323i & 0.741-9.244i & 6.811-4.409i & 6.775-2.455i & 1.895-6.195i \\
50.714-11.897i & 61.039-12.372i & 40.92-8.264i & 17.555-21.105i & -1.79-18.203i & -10.978-7.874i & -8.455-6.039i & -0.393-8.882i & 5.937-4.994i & 6.707-1.516i \\
17.767-26.218i & 40.92-8.264i & 39.815-34.457i & 29.057-9.128i & 12.417-15.267i & -1.258-13.753i & -9.649-2.78i & -6.839-4.796i & -0.372-7.125i & 4.729-4.038i \\
-4.56-21.028i & 17.555-21.105i & 29.057-9.128i & 30.716-58.342i & 22.229-9.192i & 10.159-12.663i & -4.496-10.035i & -8.457-2.476i & -6.011-4.178i & -0.35-6.237i \\
-13.17-7.22i & -1.79-18.203i & 12.417-15.267i & 22.229-9.192i & 23.275-80.655i & 17.981-9.073i & 4.731-11.657i & -3.891-8.759i & -7.348-2.181i & -5.235-3.61i \\
-10.714-4.323i & -10.978-7.874i & -1.258-13.753i & 10.159-12.663i & 17.981-9.073i & 20.08-91.295i & 13.704-8.788i & 4.403-10.855i & -3.605-8.146i & -6.819-2.037i \\
0.741-9.244i & -8.455-6.039i & -9.649-2.78i & -4.496-10.035i & 4.731-11.657i & 13.704-8.788i & 20.08-91.295i & 16.703-8.986i & 8.235-10.417i & -0.839-9.838i \\
6.811-4.409i & -0.393-8.882i & -6.839-4.796i & -8.457-2.476i & -3.891-8.759i & 4.403-10.855i & 16.703-8.986i & 20.08-91.295i & 16.703-8.986i & 8.235-10.417i \\
6.775-2.455i & 5.937-4.994i & -0.372-7.125i & -6.011-4.178i & -7.348-2.181i & -3.605-8.146i & 8.235-10.417i & 16.703-8.986i & 20.08-91.295i & 16.703-8.986i \\
1.895-6.195i & 6.707-1.516i & 4.729-4.038i & -0.35-6.237i & -5.235-3.61i & -6.819-2.037i & -0.839-9.838i & 8.235-10.417i & 16.703-8.986i & 20.08-91.295i
\end{bmatrix}$$

▲ **Step 3 of the Mathcad program.**

$$v := \begin{array}{|l} \text{for } i \in 1..N \\ \quad \begin{array}{|l} v_i \leftarrow 0 \text{ if } i \neq 2 \\ v_i \leftarrow 1 \text{ if } i = 2 \end{array} \\ v \end{array} \qquad v = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad I := Z^{-1} v$$

▲ Step 4 of the Mathcad program.

$$Zi := \frac{v_2}{I_2} \qquad Zi = 24.625 - 18.101i \qquad w := v_2 I_2$$

$$w = 0.026 - 0.019i \qquad wR := \mathrm{Re}(w) \qquad wR = 0.026$$

$$i00 := \sqrt{\frac{wR}{73}} \qquad i00 = 0.019 \qquad p := 1, 2..360$$

▲ Step 5 of the Mathcad program.

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$$b := \begin{array}{|l} \text{for } j \in 1..N \\ \quad \text{for } s \in 1..360 \\ \qquad \begin{array}{|l} dr_j \leftarrow l_{1,j} \\ dr_j \leftarrow 0 \text{ if } j = 1 \\ dr \\ a_{s,j} \leftarrow \overrightarrow{\left(dr_j \cos\left(s \frac{\pi}{180}\right)\right)} \\ A_{s,j} \leftarrow \frac{\overrightarrow{I_j e^{j\, a_{s,j}}}}{i00} \\ A_{s,j} \leftarrow \overrightarrow{\left(A_{s,j} \frac{\cos\left(\frac{\pi}{2}\sin\left(s\frac{\pi}{180}\right)\right)}{\cos\left(.00001 + s\frac{\pi}{180}\right)}\right)} \\ B_{s,j} \leftarrow \frac{\overrightarrow{I_j e^{j\, a_{s,j}}}}{i00} \\ k \leftarrow \sum_{n=1}^{N} I_n \\ C_s \leftarrow \overrightarrow{\left(\frac{k}{i00} \frac{\cos\left(\frac{\pi}{2}\cos\left(s\frac{\pi}{180}\right)\right)}{\sin\left(.00001 + s\frac{\pi}{180}\right)}\right)} \end{array} \end{array} \\ \begin{bmatrix} a \\ dr \\ A \\ B \\ C \end{bmatrix} \end{array}$$

▲ Step 6 of the Mathcad program.

$$E := \left[\sum_{n=1}^{N}\left[(b_3)^n\right]\right] \quad B := \left[\sum_{n=1}^{N}\left[(b_4)^n\right]\right]$$

▲ Step 7 of the Mathcad program.